

NO TOUCHDOWN AT ZERO POINTS OF THE PERMITTIVITY PROFILE FOR THE MEMS PROBLEM

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ABSTRACT. We study the quenching behavior for a semilinear heat equation arising in models of micro-electro mechanical systems (MEMS). The problem involves a source term with a spatially dependent potential, given by the dielectric permittivity profile, and quenching corresponds to a touchdown phenomenon. It is well known that quenching does occur. We prove that **touchdown cannot occur at zero points of the permittivity profile**. In particular, we remove the assumption of compactness of the touchdown set, made in all previous work on the subject and whose validity is unknown in most typical cases. This answers affirmatively a conjecture made in [Y. Guo, Z. Pan and M.J. Ward, *SIAM J. Appl. Math.* **66** (2005), 309–338] on the basis of numerical evidence. The result crucially depends on a new type I estimate of the quenching rate, that we establish. In addition we obtain some sufficient conditions for compactness of the touchdown set, without convexity assumption on the domain. These results may be of some qualitative importance in applications to MEMS optimal design, especially for devices such as micro-valves.

Keywords. Micro-electro mechanical systems (MEMS), touchdown, quenching points, permittivity profile, type I estimate, compactness.

AMS Classifications. Primary 35K55, 35B40, 35B44; Secondary 74K15, 74F15.

1. INTRODUCTION AND MAIN RESULTS

In this paper, we consider the problem:

$$(1.1) \quad u_t = \Delta u + f(x)(1 - u)^{-p}, \quad x \in \Omega, \quad t > 0,$$

$$(1.2) \quad u = 0, \quad x \in \partial\Omega, \quad t > 0,$$

$$(1.3) \quad u(x, 0) = 0, \quad x \in \Omega,$$

where Ω is a bounded domain of \mathbb{R}^n ($n \geq 1$), of class $C^{2+\nu}$ for some $\nu > 0$,

$$(1.4) \quad p > 0 \text{ and } f \text{ is a nonnegative, Hölder continuous function on } \overline{\Omega}, \text{ with } f \not\equiv 0.$$

A typical case of interest is the following

$$(1.5) \quad u_t = \Delta u + \lambda|x|^m(1 - u)^{-2}, \quad x \in \Omega, \quad t > 0,$$

$$(1.6) \quad u = 0, \quad x \in \partial\Omega, \quad t > 0,$$

$$(1.7) \quad u(x, 0) = 0, \quad x \in \Omega,$$

where m, λ are positive constants. This problem arises in the study of modeling the dynamic deflection of an elastic membrane inside a micro-electro mechanical system (MEMS).

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The full model is

$$(1.8) \quad \epsilon u_{tt} + u_t = \Delta u + \frac{\lambda g(x)}{(1-u)^2 \left(1 + \alpha \int_{\Omega} \frac{1}{1-u} dx\right)^2}, \quad x \in \Omega, \quad t > 0,$$

where ϵ is the ratio of the interaction due to the inertial and damping terms, λ is proportional to the applied voltage, u is the deflection of the membrane (the natural physical dimension being thus $n = 2$). The function $g(x)$, called the permittivity profile, represents varying dielectric properties of the membrane. One of physically suggested dielectric profiles is the power-law profile $g(x) = |x|^m$ with $m > 0$. The integral in (1.8) arises due to the fact that the device is embedded in an electrical circuit with a capacitor of fixed capacitance. The parameter α denotes the ratio of this fixed capacitance to a reference capacitance of the device. As for the initial condition (1.3), it means that the membrane has initially no deflection, the voltage being switched on at $t = 0$. For the details of background and derivation of this model, we refer the reader to [22, 23, 6].

The case when $\epsilon = 0$ is studied in [14, 15, 13] for $\alpha > 0$ and f a constant. We shall here concentrate on the case when $\epsilon = \alpha = 0$ (so that there is no capacitor in the circuit) and f is nonconstant. It has been studied extensively for past years, see, e.g., the works [11, 18, 10, 16, 17, 20, 26]. For the study of stationary solutions, we refer to [21, 12, 1, 9, 2, 3, 19, 20, 26].

By the standard parabolic theory, there exists a unique classical solution of (1.1)-(1.3) in a short time interval. Also, by the strong maximum principle, we have $u > 0$ in Ω for $t > 0$. Moreover, the solution u of (1.1)-(1.3) can be continued as long as $\max_{x \in \bar{\Omega}} u(x, t) < 1$. We shall let $[0, T)$ be the maximal existence time interval of u , where $T \leq \infty$. If $T < \infty$, then quenching occurs in finite time, i.e.

$$\limsup_{t \rightarrow T^-} \left\{ \max_{x \in \bar{\Omega}} u(x, t) \right\} = 1.$$

It is well known (see, e.g., [4, 20] and the references therein) that the solution of (1.1)-(1.3) quenches in finite time when λ is sufficiently large. A point $x = x_0$ is a *quenching point* if there exists a sequence $\{(x_n, t_n)\}$ in $\bar{\Omega} \times (0, T)$ such that

$$x_n \rightarrow x_0, \quad t_n \uparrow T \quad \text{and} \quad u(x_n, t_n) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

The set of all quenching points is called the quenching set, denoted by \mathcal{Q} . In the context of MEMS, quenching corresponds to a touchdown phenomenon.

Note that in the typical case of (1.5), there is no source at $x = 0$ due to the spatially dependent coefficient $|x|^m$. A long-standing open problem, even in one space dimension, is to determine whether or not $x = 0$ is a quenching point. More generally, for problem (1.1)-(1.3), the question is whether a point x_0 such that $f(x_0) = 0$ can be a quenching point. In [10, 16], under the assumption that the quenching set is a compact subset of Ω , it is shown that x_0 is not a quenching point if $f(x_0) = 0$. On the other hand, the compactness assumption was proved in [16] by adapting a moving plane argument from [7, 11] when f is constant or, more generally, when f is nonincreasing as one approaches the boundary. However, for the typical problem (1.5)-(1.7) it is unknown whether the quenching set is compact. Actually, supported by numerical evidence provided in [18], the following conjecture was made (see [10, 16, 4]):

Conjecture. *The point $x = 0$ is not a quenching point for problem (1.5)-(1.7).*

In the present paper, we give an affirmative answer to this conjecture, as well as for the case of general f , in any space dimension. Our main result is the following.

Theorem 1. *Assume (1.4) and let the solution u of problem (1.1)-(1.3) be such that $T < \infty$. If $x_0 \in \Omega$ is such that $f(x_0) = 0$, then x_0 is not a quenching point.*

In particular, as a special case, we have that 0 is not a quenching point for problem (1.5)-(1.7). Actually, we have been able to answer this question *independently* of the compactness issue of quenching set. In fact, as a key-step – of independent interest – to the proof of Theorem 1, we prove the following estimate, which in particular guarantees that the quenching rate is of *type I* on any compact subset of Ω . In what follows, we denote

$$\delta(x) = \text{dist}(x, \partial\Omega), \quad x \in \overline{\Omega},$$

the function distance to the boundary.

Theorem 2. *Assume (1.4) and let the solution u of problem (1.1)-(1.3) be such that $T < \infty$. Then there exists a constant $\gamma > 0$ (independent of x, t) such that*

$$(1.9) \quad 1 - u(x, t) \geq \gamma \delta(x) (T - t)^{1/(p+1)}, \quad x \in \Omega, \quad 0 < t < T.$$

Theorem 2 will be proved via a nontrivial modification of the Friedman-McLeod method ([7], see also [11]). Once Theorem 2 is proved, Theorem 1 will be deduced by constructing a suitable local supersolution.

The compactness of the quenching set remains an open question. In particular, we do not know if Theorem 1 remains true if $f(x_0) = 0$ with $x_0 \in \partial\Omega$ (in other words, can a zero *boundary* point of the permittivity profile be a quenching point?). As mentioned above, this cannot occur if we assume in addition that f is nonincreasing as one approaches the boundary. Actually, as a consequence of Theorem 2 and of suitable comparison arguments, we have been able to obtain two further criteria for the quenching set to be compact. We note that, unlike in the aforementioned criterion, we do not require any convexity of the domain Ω .

Theorem 3. *Assume (1.4) and let the solution u of problem (1.1)-(1.3) be such that $T < \infty$. Assume either*

$$(1.10) \quad 0 < p < 1,$$

or

$$(1.11) \quad f(x) = o(\delta^{p+1}(x)) \quad \text{as } \delta(x) \rightarrow 0.$$

Then quenching does not occur near the boundary, i.e. $\mathcal{Q} \subset \Omega$.

Going back to MEMS modeling, it seems that Theorem 1 and the case of (1.11) in Theorem 3 may be of some importance in applications, at least from the qualitative point of view (see [18, 4] for more details). This is especially true for particular devices of MEMS type such as micro-valves, where the touchdown behavior is explicitly exploited, since the touchdown or quenching set then corresponds to the lid or closing area of the valve. As a consequence of our results, we see that the latter has to be part of the positive set of the function f . The choice of f , through an appropriate repartition of the dielectric coating, can thus be used in the optimal design of the microvalve. In this respect, it would

be desirable to gain further information about the structure of the quenching set, but this seems a difficult mathematical problem for nonconstant f , even in one space dimension.

Remark. We point out that Theorems 1–3 still hold if we replace as in [20] the zero initial data by a nonnegative C^2 function u_0 such that $u_0 < 1$ in $\overline{\Omega}$, $\Delta u_0 + f(x)(1 - u_0)^{-p} \geq 0$ in Ω and $u_0 = 0$ on $\partial\Omega$. Indeed, this assumption guarantees that $u_t > 0$ and the proofs can then be modified in a straightforward way.

2. PROOF OF THEOREM 2

2.1. General strategy and basic computation. When the compactness of the quenching set is known, type I estimates can be proved by means of the maximum principle applied, in a strict subdomain of Ω , to the well-known auxiliary function (cf. [7, 11]):

$$J(x, t) := u_t - \varepsilon(1 - u)^{-p},$$

where ε is a small positive constant. In the present situation, the possible noncompactness of the quenching set prevents one to verify that $J \geq 0$ on the boundary of any subdomain of Ω and the method is not directly applicable.

To overcome this, our basic idea is to consider a modified function J as follows:

$$(2.1) \quad J(x, t) = u_t - \varepsilon a(x)h(u),$$

where $a(x)$ is an auxiliary function such that $a = 0$ on $\partial\Omega$, hence also $J = 0$. The construction is delicate and requires specific properties for a , which will be given later. As for the function $h(u)$, it will be a perturbation of the nonlinearity, namely

$$(2.2) \quad h(u) = (1 - u)^{-p} + (1 - u)^{-q}, \quad 0 < q < p.$$

Before specializing, we first present the basic computations.

Lemma 2.1. *Let J, h be given by (2.1)-(2.2), where $a \in C^2(\Omega)$ is a nonnegative function. Then*

$$(2.3) \quad J_t - \Delta J - pf(x)(1 - u)^{-p-1}J = \varepsilon R,$$

where

$$(2.4) \quad R = (p - q)a(x)f(x)(1 - u)^{-p-q-1} + ah''(u)|\nabla u|^2 + 2h'(u)\nabla a \cdot \nabla u + h(u)\Delta a.$$

Moreover, $h'' > 0$ and, at any point $x \in \Omega$ such that $a(x) > 0$, we have

$$(2.5) \quad R \geq \underbrace{(p - q)a(x)f(x)(1 - u)^{-p-q-1}}_{\mathcal{T}_1} + \underbrace{h(u)\Delta a}_{\mathcal{T}_2} - \underbrace{\frac{h'^2(u)|\nabla a|^2}{ah''(u)}}_{\mathcal{T}_3}.$$

Proof. We compute

$$\begin{aligned} J_t &= u_{tt} - \varepsilon a(x)h'(u)u_t \\ \nabla J &= \nabla u_t - \varepsilon(a(x)h'(u)\nabla u + h(u)\nabla a(x)) \\ \Delta J &= \Delta u_t - \varepsilon(a(x)h'(u)\Delta u + a(x)h''(u)|\nabla u|^2 + 2h'(u)\nabla a(x) \cdot \nabla u + h(u)\Delta a(x)). \end{aligned}$$

Setting $g(u) = (1 - u)^{-p}$ and omitting the variables x, u without risk of confusion, we get

$$\begin{aligned} J_t - \Delta J &= (u_t - \Delta u)_t - \varepsilon ah'(u_t - \Delta u) + \varepsilon(ah''|\nabla u|^2 + 2h'\nabla a \cdot \nabla u + h\Delta a), \\ &= f(x)g'u_t - \varepsilon f(x)ah'g + \varepsilon(ah''|\nabla u|^2 + 2h'\nabla a \cdot \nabla u + h\Delta a). \end{aligned}$$

Using $u_t = J + \varepsilon ah$, we have

$$J_t - \Delta J - f(x)g'J = \varepsilon R,$$

where

$$R = f(x)a(g'h - h'g) + ah''|\nabla u|^2 + 2h'\nabla a \cdot \nabla u + h\Delta a.$$

On the other hand, we have

$$\begin{aligned} g'h - h'g &= p(1-u)^{-p-1}((1-u)^{-p} + (1-u)^{-q}) \\ &\quad - (1-u)^{-p}(p(1-u)^{-p-1} + q(1-u)^{-q-1}) \\ &= (p-q)(1-u)^{-p-q-1}, \end{aligned}$$

hence (2.4). Finally, since $h'' > 0$, for all $x \in \Omega$ such that $a(x) > 0$, we may write

$$R = (p-q)af(x)(1-u)^{-p-q-1} + h\Delta a + ah''\left[|\nabla u|^2 + 2\frac{h'\nabla a \cdot \nabla u}{ah''}\right],$$

hence (2.5). \square

2.2. Construction of the function $a(x)$. We see that, in order to guarantee $R \geq 0$, the (negative) term \mathcal{T}_3 on the RHS of (2.5) must be absorbed by a positive contribution coming either:

- from the term \mathcal{T}_1 (generated by the perturbation in (2.2)), provided $f(x) > 0$; or
- from the term \mathcal{T}_2 , provided $\Delta a(x) > 0$.

Since $a(x)$ is nonnegative and vanishes at the boundary, we cannot have $\Delta a > 0$ everywhere. Actually, we shall consider functions $a(x)$ which are positive in Ω and suitably convex everywhere except on a small ball B , where f is uniformly positive. Also, it will be necessary to split the parabolic cylinder $\Omega \times (T/2, T)$ into suitable sub-regions, taking into account the “large” and “small” parts of the function $u(x, t)$.

The following essential lemma gives the construction of the appropriate function $a(x)$.

Lemma 2.2. *Let h be given by (2.2). Let $x_0 \in \Omega, \rho > 0$ with $\overline{B}(x_0, \rho) \subset \Omega$, and denote the open set*

$$\Omega_{x_0, \rho} = \Omega \setminus \overline{B}(x_0, \rho).$$

Then there exists a function $a \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ with the following properties:

$$(2.6) \quad hh''a\Delta a - h'^2|\nabla a|^2 \geq 0 \quad \text{for all } x \in \overline{\Omega_{x_0, \rho}} \text{ and all } 0 \leq u < 1,$$

$$(2.7) \quad C_1\delta^{p+1}(x) \leq a(x) \leq C_2\delta^{p+1}(x) \quad \text{for all } x \in \overline{\Omega},$$

for some constants $C_1, C_2 > 0$.

Proof. For $0 < u < 1$, computing

$$(2.8) \quad h'(u) = p(1-u)^{-p-1} + q(1-u)^{-q-1},$$

$$(2.9) \quad h''(u) = p(p+1)(1-u)^{-p-2} + q(q+1)(1-u)^{-q-2},$$

it follows that

$$\begin{aligned} hh'' &= p(p+1)(1-u)^{-2p-2} + (p(p+1) + q(q+1))(1-u)^{-p-q-2} \\ &\quad + q(q+1)(1-u)^{-2q-2} \end{aligned}$$

$$\geq \frac{p+1}{p} \left[p^2(1-u)^{-2p-2} + 2pq(1-u)^{-p-q-2} + q^2(1-u)^{-2q-2} \right],$$

where we used $p(p+1) + q(q+1) > 2(p+1)q$ due to $0 < q < p$. Therefore, we have

$$(2.10) \quad hh'' \geq \frac{p+1}{p} (h')^2 \quad \text{for } 0 \leq u < 1.$$

Next we introduce a suitable harmonic function ϕ , namely, the unique solution of the problem

$$\begin{aligned} \Delta\phi &= 0, & x &\in \Omega_{x_0, \rho}, \\ \phi &= 0, & x &\in \partial\Omega, \\ \phi &= 1, & x &\in \partial B(x_0, \rho). \end{aligned}$$

The function ϕ is smooth and, by the strong maximum principle and the Hopf Lemma, we have $0 < \phi < 1$ in $\Omega_{x_0, \rho}$ and

$$(2.11) \quad c_1\delta(x) \leq \phi(x) \leq c_2\delta(x), \quad x \in \overline{\Omega_{x_0, \rho}}$$

for some positive constants c_1, c_2 . Then we set

$$(2.12) \quad a(x) = \phi^{p+1}(x), \quad x \in \overline{\Omega_{x_0, \rho}}.$$

Since $a \in C^2(\Omega_{x_0, \rho})$, the boundary $\partial B(x_0, \rho)$ is smooth and $a = 1$ on $\partial B(x_0, \rho)$, the function a can be extended in $B(x_0, \rho)$ in such a way that $a \in C^2(\Omega)$ and $a > 0$ in Ω .

On the other hand, on $\Omega_{x_0, \rho}$, we compute:

$$\nabla a = (p+1)\phi^p \nabla \phi, \quad \Delta a = (p+1)[\phi^p \Delta \phi + p\phi^{p-1}|\nabla \phi|^2] = p(p+1)\phi^{p-1}|\nabla \phi|^2,$$

hence

$$(2.13) \quad a\Delta a = \frac{p}{p+1} |\nabla a|^2 \quad \text{on } \Omega_{x_0, \rho}.$$

Combining (2.10) and (2.13), we get (2.6). Property (2.7) follows from (2.11), (2.12) and $a > 0$ in Ω . \square

Before going further, let us recall the following useful lower bound on u_t .

Lemma 2.3. *There exists a constant $c_0 > 0$ such that*

$$(2.14) \quad u_t \geq c_0\delta(x) \quad \text{on } \Omega \times [T/2, T].$$

Proof. Although the proof of the lemma is standard (cf. [7, 11, 16]), we provide a proof here for completeness. Setting $v = u_t$, we see that v satisfies

$$\begin{aligned} v_t &= \Delta u + pf(x)(1-u)^{-p-1}v, & x &\in \Omega, \quad 0 < t < T, \\ v &= 0, & x &\in \partial\Omega, \quad 0 < t < T, \\ v(x, 0) &= f(x), & x &\in \Omega, \end{aligned}$$

so that $v = u_t \geq 0$ in $Q_T := \Omega \times (0, T)$ by the maximum principle.

Applying the maximum principle again, we deduce that $u_t \geq z$ in Q_T , where z is the solution of the heat equation in Q_T , with zero boundary condition and initial condition $z(\cdot, 0) = f$. Since z satisfies the estimate (2.14) in virtue of the Hopf lemma and the strong maximum principle, so does u_t . \square

2.3. Proof of Theorem 2.

Step 1. Preparations. Since $f \geq 0$ and $f \not\equiv 0$ is continuous, we may pick a point $x_0 \in \Omega$ and $\rho > 0$ such that $B(x_0, 2\rho) \subset \Omega$ and

$$(2.15) \quad \sigma_1 := \inf_{x \in \overline{B}(x_0, \rho)} f(x) > 0.$$

We then take $a \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ as given by Lemma 2.2, and define J by

$$J(x, t) = u_t - \varepsilon a(x)h(u),$$

with $\varepsilon > 0$ to be fixed later and

$$(2.16) \quad h(u) = (1 - u)^{-p} + (1 - u)^{-q}, \quad 0 \leq u < 1, \quad \text{where } q = p/2$$

(any choice of $q \in (0, p)$ would do). Note that

$$(2.17) \quad \sigma_2 := \inf_{x \in \overline{B}(x_0, \rho)} a(x) > 0.$$

Next, we split the cylinder $\Sigma := \Omega \times (T/2, T)$ into three sub-regions as follows:

$$\Sigma_1 = (\Omega \setminus \overline{B}(x_0, \rho)) \times (T/2, T),$$

$$\Sigma_2^\eta = \{(x, t) \in \overline{B}(x_0, \rho) \times (T/2, T); u(x, t) \geq 1 - \eta\}$$

and

$$\Sigma_3^\eta = \{(x, t) \in \overline{B}(x_0, \rho) \times (T/2, T); u(x, t) < 1 - \eta\},$$

where the number $\eta \in (0, 1)$ will be specified later on.

Step 2. Parabolic inequality for J in the sub-regions Σ_1 and Σ_2^η . It follows from properties (2.5) in Lemma 2.1 and (2.6) in Lemma 2.2, along with $a > 0$, $f \geq 0$ in Ω and $h'' > 0$, that

$$(2.18) \quad J_t - \Delta J - pf(x)(1 - u)^{-p-1}J \geq 0 \quad \text{in } \Sigma_1.$$

Next, in view of (2.16) and (2.8), we have

$$|h\Delta a| \leq C_3(1 - u)^{-p}, \quad |h'\nabla a| \leq C_3(1 - u)^{-p-1} \quad \text{in } \Sigma$$

for some positive constant C_3 independent of ε, η . Also, from (2.9) and (2.17) we get

$$ah'' \geq \sigma_2 p(p+1)(1 - u)^{-p-2} \quad \text{in } \overline{B}(x_0, \rho) \times (0, T).$$

Consequently, recalling the definition of R in Lemma 2.1, it follows from (2.5), (2.15) and (2.17) that

$$\begin{aligned} (1 - u)^{p+q+1}R &\geq (p - q)f(x)a + h\Delta a(1 - u)^{p+q+1} - \frac{(h'|\nabla a|)^2}{ah''}(1 - u)^{p+q+1} \\ &\geq (p - q)\sigma_1\sigma_2 - C_4(1 - u)^{q+1} \geq (p - q)\sigma_1\sigma_2 - C_4\eta^{q+1} \quad \text{in } \Sigma_2^\eta, \end{aligned}$$

for some positive constant C_4 independent of ε, η . Owing to (2.3), we may thus choose $\eta \in (0, 1)$ small, independent of ε , such that

$$(2.19) \quad J_t - \Delta J - pf(x)(1 - u)^{-p-1}J \geq 0 \quad \text{in } \Sigma_2^\eta.$$

Step 3. Control of J on Σ_3^η and conclusion. Now that η has been fixed, using (2.7), (2.14) and (2.16), we may choose $\varepsilon > 0$ small enough, so that

$$(2.20) \quad J \geq \delta(x) \left[c_0 - 2C_2\varepsilon\delta^p(x)(1 - u)^{-p} \right] \geq \delta(x) \left[c_0 - 2C_2\varepsilon\delta^p(x)\eta^{-p} \right] \geq 0 \quad \text{in } \Sigma_3^\eta$$

and

$$(2.21) \quad J(x, T/2) \geq \delta(x) \left[c_0 - 2C_2 \varepsilon \delta^p(x) (1 - \|u(\cdot, T/2)\|_\infty)^{-p} \right] \geq 0 \quad \text{in } \overline{\Omega},$$

where c_0 is the constant in Lemma 2.3 and C_1, C_2 are the constants in (2.7). Observe that, as a consequence of (2.20) and $\Sigma = \Sigma_1 \cup \Sigma_2^\eta \cup \Sigma_3^\eta$, we have

$$(2.22) \quad \{(x, t) \in \Sigma; J(x, t) < 0\} \subset \Sigma_1 \cup \Sigma_2^\eta.$$

Also, since $a = 0$ on $\partial\Omega$, we have

$$(2.23) \quad J = 0 \quad \text{on } \partial\Omega \times (T/2, T).$$

On the other hand, by standard parabolic regularity, we observe that

$$J \in C^{2,1}(\Sigma) \cap C(\overline{\Omega} \times [T/2, T)).$$

It follows from (2.18)-(2.19), (2.21)-(2.23) and the maximum principle (see e.g. [25, Proposition 52.4 and Remark 52.11(a)]) that

$$J \geq 0 \quad \text{in } \Sigma.$$

Then, for $T/2 < t < s < T$ and $x \in \Omega$, taking (2.7) into account, an integration in time gives

$$\begin{aligned} (1 - u(x, t))^{p+1} &\geq (1 - u(x, s))^{p+1} + C_1 \varepsilon (p+1) \delta^{p+1}(x) (s - t) \\ &\geq C_1 \varepsilon (p+1) \delta^{p+1}(x) (s - t). \end{aligned}$$

Letting $s \rightarrow T$, we finally deduce (1.9) in Σ , hence in $\Omega \times (0, T)$. Note that the constants C_1, ε are independent of x, t , so is the constant γ in (1.9). \square

3. PROOF OF THEOREM 1

With the type I estimate (1.9) of Theorem 2 at hand, the proof is done via a suitable local comparison function. Let $x_0 \in \Omega$ be such that $f(x_0) = 0$ and take $b_0 \in (0, 1)$ small such that

$$(3.1) \quad \overline{B}(x_0, 2b_0) \subset \Omega.$$

We consider the following function

$$w(x, t) := 1 - A[\phi(x) + (T - t)]^{1/(p+1)} \quad \text{in } \overline{B}(x_0, b) \times [0, T],$$

where

$$\phi(x) := \kappa b^2 \left(1 - \frac{|x - x_0|^2}{b^2} \right)^2.$$

Here, $\kappa \in (0, 1)$ and $b \in (0, b_0)$ are constants to be chosen later and A is a fixed positive constant such that $A \leq \gamma b_0$ and $A \leq (1 + T)^{-1/(p+1)}$ (where γ is the constant given in (1.9)). Note that

$$w(x, 0) = 1 - A[\phi(x) + T]^{1/(p+1)} \geq 0 \quad \text{for } x \in \overline{B}(x_0, b)$$

and

$$\begin{aligned} w(x, t) &= 1 - A(T - t)^{1/(p+1)} \\ &\geq 1 - \gamma \delta(x) (T - t)^{1/(p+1)} \geq u(x, t) \quad \text{for } (x, t) \in \partial B(x_0, b) \times (0, T), \end{aligned}$$

due to (3.1), $A \leq \gamma b_0$ and (1.9).

We compute, in $B(x_0, b) \times (0, T)$,

$$\begin{aligned}
& w_t - \Delta w - f(x)(1-w)^{-p} \\
&= \frac{A}{p+1} [\phi(x) + (T-t)]^{-1+\frac{1}{p+1}} + \frac{A}{p+1} [\phi(x) + (T-t)]^{-1+\frac{1}{p+1}} \Delta \phi \\
&\quad - \frac{Ap}{(p+1)^2} [\phi(x) + (T-t)]^{-2+\frac{1}{p+1}} |\nabla \phi|^2 - f(x) A^{-p} [\phi(x) + (T-t)]^{-p/(p+1)} \\
&= \frac{A}{p+1} [\phi(x) + (T-t)]^{-p/(p+1)} \\
&\quad \times \left\{ 1 + \Delta \phi - \frac{p}{p+1} [\phi(x) + (T-t)]^{-1} |\nabla \phi|^2 - (p+1) A^{-p-1} f(x) \right\},
\end{aligned}$$

hence

$$\begin{aligned}
(3.2) \quad & w_t - \Delta w - f(x)(1-w)^{-p} \\
& \geq \frac{A}{p+1} [\phi(x) + (T-t)]^{-p/(p+1)} \left\{ 1 + \Delta \phi - \frac{p}{p+1} \frac{|\nabla \phi|^2}{\phi} - (p+1) A^{-p-1} f(x) \right\}.
\end{aligned}$$

Moreover, in $B(x_0, b)$ we have

$$\begin{aligned}
\nabla \phi(x) &= -4\kappa \left(1 - \frac{|x-x_0|^2}{b^2} \right) (x-x_0), \\
\Delta \phi(x) &= -4n\kappa \left(1 - \frac{|x-x_0|^2}{b^2} \right) + 8\kappa \left(\frac{|x-x_0|}{b} \right)^2 \geq -4n\kappa \\
\frac{|\nabla \phi(x)|^2}{\phi(x)} &= 16\kappa \left(\frac{|x-x_0|}{b} \right)^2 \leq 16\kappa.
\end{aligned}$$

Now, since $f(x_0) = 0$, we may choose $b > 0$ small enough so that

$$(p+1) A^{-p-1} \sup_{x \in B(x_0, b)} f(x) \leq \frac{1}{3}.$$

Then we can choose $\kappa = \kappa(n) \in (0, 1)$ small enough such that

$$\Delta \phi(x) \geq -\frac{1}{3}, \quad \frac{|\nabla \phi(x)|^2}{\phi(x)} \leq \frac{1}{3} \quad \text{for all } x \in B(x_0, b).$$

Therefore, $w_t - \Delta w - f(x)(1-w)^{-p} \geq 0$ in $B(x_0, b) \times (0, T)$, and it follows from the comparison principle that $u \leq w$ in $B(x_0, b) \times (0, T)$. Since $\min_{\overline{B}(x_0, b/2)} \phi > 0$, this implies that $x = x_0$ is not a quenching point and the theorem is proved. \square

4. PROOF OF THEOREM 3

(i) **Case of (1.10).** Assume without loss of generality that $0 \neq x_0 \in \partial\Omega$ with $B(0, |x_0|) \cap \Omega = \emptyset$. This (exterior ball condition) is possible due to the assumption

that $\partial\Omega \in C^{2+\nu}$. We look for a supersolution of the form $z(x) = 1 - C(d - r)^\beta$, $r = |x|$, with $\beta > 1$, $C > 0$, $d > |x_0|$ to be chosen. For $0 < r < d$, we have:

$$\begin{aligned} z_r &= \beta C(d - r)^{\beta-1} \\ z_{rr} &= -\beta(\beta - 1)C(d - r)^{\beta-2}. \end{aligned}$$

Set $\omega := \Omega \cap \{x \in \mathbb{R}^n; |x_0| < |x| < d\}$. Choosing $\beta = 2/(p+1) > 1$, so that $\beta - 2 = -\beta p$, we compute in ω :

$$\begin{aligned} -\Delta z - f(x)(1 - z)^{-p} &= \beta(\beta - 1)C(d - |x|)^{\beta-2} - \frac{\beta C(n - 1)(d - |x|)^{\beta-1}}{|x|} \\ &\quad - f(x)[C(d - |x|)^\beta]^{-p} \\ &= C(d - |x|)^{-\beta p} \left[\beta(\beta - 1) - \frac{\beta(n - 1)(d - |x|)}{|x|} - C^{-p-1}f(x) \right]. \end{aligned}$$

Next taking $d = d(p, n, |x_0|) > |x_0|$ close to $|x_0|$ and $C = C(p, \|f\|_\infty) > 0$ large, we then have, in ω :

$$-\Delta z - f(x)(1 - z)^{-p} \geq C(d - |x|)^{-\beta p} \left[\beta(\beta - 1) - \frac{\beta(n - 1)(d - |x_0|)}{|x_0|} - C^{-p-1}\|f\|_\infty \right] \geq 0.$$

By taking d possibly closer to $|x_0|$, we have

$$z(x) \geq 1 - C(d - |x_0|)^\beta \geq 0 \quad \text{in } \bar{\omega},$$

hence also $z(x) \geq u(x, t) = 0$ for $x \in \partial\omega \cap \{|x| < d\} = \partial\omega \cap \partial\Omega$ and $t \in (0, T)$. Since on the other hand $z(x) = 1 > u(x, t)$ for $x \in \partial\omega \cap \{|x| = d\}$ and $0 < t < T$, we deduce from the comparison principle that $z \geq u$ on $\omega \times (0, T)$. Therefore x_0 is not a quenching point.

(ii) **Case of (1.11).** The proof relies on estimate (1.9) and is a modification of that of Theorem 1. Let $\Omega_\eta = \{x \in \Omega; \delta(x) < \eta\}$. There exists $\eta_0 > 0$ such that Ω_η is a smooth bounded domain for all $\eta \in (0, \eta_0)$, due to Ω being a smooth domain. We have $\partial\Omega_\eta = \partial\Omega \cup \Gamma_\eta$, where $\Gamma_\eta = \{x \in \Omega; \delta(x) = \eta\}$.

Let ψ_η be the unique solution of the problem

$$\begin{aligned} \Delta\psi_\eta &= 0, \quad x \in \Omega_\eta, \\ \psi_\eta &= 1, \quad x \in \partial\Omega, \\ \psi_\eta &= 0, \quad x \in \Gamma_\eta. \end{aligned}$$

The function ψ_η is smooth and, by the strong maximum principle, we have $0 < \psi_\eta < 1$ in Ω_η . Letting

$$\phi = k\psi_\eta^2,$$

we consider the function

$$w(x, t) := 1 - \gamma\eta[\phi(x) + (T - t)]^{1/(p+1)} \quad \text{in } \bar{\Omega}_\eta \times [0, T],$$

where $k \in (0, 1)$ and $\eta \in (0, \eta_0)$ are constants to be chosen later and γ is the constant given in (1.9). First assuming $\eta \leq \eta_1 := \min(\eta_0, \gamma^{-1}(1 + T)^{-1/(p+1)})$, we have

$$w(x, 0) = 1 - \gamma\eta[\phi(x) + T]^{1/(p+1)} \geq 0 \quad \text{in } \bar{\Omega}_\eta,$$

along with $w(x, t) \geq 0$ on $\partial\Omega \times (0, T)$ and, by (1.9),

$$w(x, t) = 1 - \gamma\eta(T - t)^{1/(p+1)} = 1 - \gamma\delta(x)(T - t)^{1/(p+1)} \geq u(x, t) \quad \text{on } \Gamma_\eta \times (0, T).$$

Formula (3.2) remains valid, with A replaced by $\gamma\eta$, and moreover we have

$$\Delta\phi = 2k|\nabla\psi_\eta|^2 \quad \text{and} \quad \frac{|\nabla\phi|^2}{\phi} = 4k|\nabla\psi_\eta|^2.$$

Therefore,

$$\begin{aligned} & w_t - \Delta w - f(x)(1 - w)^{-p} \\ & \geq \frac{\gamma\eta}{p+1} [\phi(x) + (T - t)]^{-p/(p+1)} \left\{ 1 - \frac{4kp}{p+1} |\nabla\psi_\eta|^2 - (p+1)(\gamma\eta)^{-p-1} f(x) \right\} \end{aligned}$$

in $\Omega_\eta \times [0, T)$. Now, by assumption (1.11), we may choose $\eta \in (0, \eta_1)$ small enough so that

$$(p+1)(\gamma\eta)^{-p-1} \sup_{x \in \Omega_\eta} f(x) \leq \frac{1}{2}.$$

Then we can choose $k = k(\eta) > 0$ small enough so that

$$4k|\nabla\psi_\eta|^2 \leq \frac{1}{2} \quad \text{for all } x \in \Omega_\eta.$$

Therefore, $w_t - \Delta w - f(x)(1 - w)^{-p} \geq 0$ in $\Omega_\eta \times (0, T)$, and it follows from the comparison principle that $u \leq w$ in $\Omega_\eta \times (0, T)$. Since $\min_{\overline{\Omega}_{\eta/2}} \psi_\eta > 0$, this guarantees that no quenching occurs near the boundary. \square

Remark. Although it is not clear if Theorem 3(i) has a direct relation with this fact, it is interesting to recall that, when f is constant, quenching for problem (1.1)-(1.3) is incomplete (in the sense of existence of a suitable weak continuation after $t = T$) if and only if $0 < p < 1$ (cf. [24, 5, 8]).

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